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THE SUSPENDED ELASTIC CABLE UNDER THE ACTION OF CONCENTRATED VERTICAL LOADS

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Abstract—This investigation treats the static response of a single elastic cable which is suspended between two points that are not necessarily at the same level. The cable is loaded by its self-weight and any number of concentrated vertical loads which may be arbitrarily placed along its length. The analysis presented uses a Lagrangian approach. For the strained cable profile, the tension and displacements are given as functions of a single Lagrangian co-ordinate. A specific application of the general analysis is made and compared with a simple experiment.

INTRODUCTION

The structural use of suspended cables has long been a topic of interest. Well known structural uses of suspended cable systems include electrical transmission lines, suspension bridges and cable roofs. Pugsley[1] gives some interesting historical perspectives on the use of suspended cables in his book on suspension bridge theory. The Subcommittee on Cable-Suspended Roof Construction of the A.S.C.E.[2] has recently compiled a review of published work on the behaviour of suspended cable systems (up to 1971).

The present paper treats the *static response of a single cable* suspended between two rigid supports which are not necessarily at the same level. The cable itself is of constant cross-sectional area when unloaded, and is composed of a homogeneous material which is linearly elastic. Stresses within the cable are assumed to be axial and tensile, with a uniform distribution with respect to cable cross-section. Loading of the cable is provided by the cable self-weight and by any number of concentrated vertical loads which may be arbitrarily distributed along the cable's length. The tension within the cable and the associated displacements under such loadings are sought. This is a *general problem* of some practical importance.

The major portion of previous work done on the static response of a single cable has been either approximate (i.e. entailing simplifications of the problem), or by means of a direct numerical approach (such as finite element). No single approximate method can solve the problem completely: direct numerical approaches typically encounter convergence difficulties. Consequently an *exact* analytical solution is of value—exact in this context meaning the problem as outlined in the preceding paragraph without additional assumptions.

Work of an exact nature on the static response of a single cable is contained in a book by Routh[3] where the solution of the symmetrically-suspended elastic cable under self-weight alone is presented (the so-called elastic caternary). Feld[4] extended Routh's analysis to the unsymmetrically-suspended elastic catenary. Unfortunately, the co-ordinate system used in both analyses makes application of the results awkward.

In what follows, a different fomulation is used and the general problem (involving any number of concentrated vertical loads) tackled. Section 1 contains this formulation which features two Lagrangian co-ordinates, each corresponding to the length of cable between the support at the origin and some particle point—the first being associated with the unstrained profile while the second is associated with the strained profile. Explicit expressions for the tension within the cable and the rectangular co-ordinates describing the strained profile are derived as functions of a single independent variable, the *Lagrangian co-ordinate associated with the unstrained profile.* These expressions contain, as unknowns, the horizontal and vertical reactions of the support at the origin. These quantities can be determined by solving an elementary transcendental equation simultaneously with an algebraic equation to afford the complete solution to the general problem.

The appropriateness of the co-ordinate system employed is demonstrated by the application of the results found in Section 1 to a specific problem involving a single concentrated load in

Section 2. By way of comparison, a simple experiment is performed. Agreement between theoretical and experimental values is good, with both sets of results exhibiting the *geometric non-linear stiffening* that characterizes static cable response.

1. FORMULATION: DERIVATION OF THE GENERAL SOLUTION

We now formulate the general suspended cable problem and seek solutions having, as their independent variable, the Lagrangian co-ordinate associated with the unstrained cable.

In formulating the problem, we first turn to the *geometry* of the cable and the *co-ordinate systems* used to describe this geometry. Consider a cable suspended between the two fixed points P_0 and P_{N+1} which have plane rectangular cartesian co-ordinates (0, 0) and (l, h), respectively. Thus the span of the cable is l and the relative vertical displacement of the end-points is h. The unstrained length of the cable between the end-points is L_0. A particle point P on the cable has Lagrangian co-ordinate s in the unstrained profile, i.e. the length of the cable between P_0 and P is s when the cable is unloaded. When the cable is loaded by the self-weight W and N concentrated loads $F_n (n = 1, 2, \ldots, N)$, this particle point moves to occupy its new position in the strained profile. The concentrated vertical loads are applied at the points $P_n (n = 1, 2, \ldots, N)$ which have Lagrangian co-ordinates $s_n (n = 1, 2, \ldots, N)$ in the unstrained profile. The particle point P in the strained profile has plane rectangular cartesian co-ordinates x and y and Lagrangian co-ordinate p (Fig. 1).

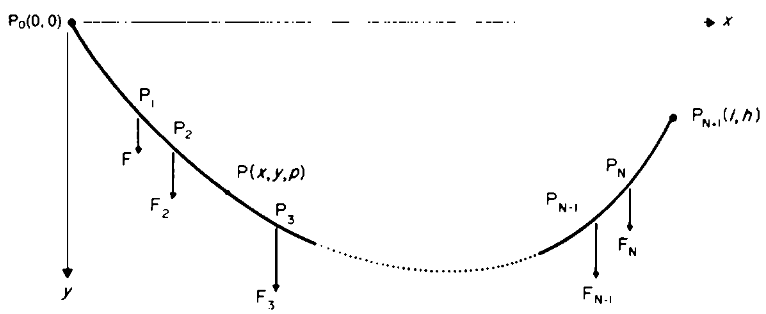

Fig. 1. Co-ordinates for the strained cable profile.

Since the cable lies wholly in the xy-plane, the following *geometric constraint* must be satisfied if P is to lie on the strained profile:

$$\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 = 1. \tag{1}$$

Equation (1) holds throughout the strained cable profile except at the points of application of the concentrated loads $P_n (n = 1, 2, \ldots, N)$ where the slope of the profile is discontinuous. These discontinuities are a feature of the present problem and, as such, frequently restrict the range of validity of the governing equations in our formulation. To avoid repetition of the full statement of this restriction we will hereafter refer to equations as holding on the *continuous strained profile*, that is, the strained cable profile including the points P_0, P_{N+1} but excluding the points $P_n (n = 1, 2, \ldots, N)$.

If the mass-density of the cable is constant and its cross-sectional area in the unstrained profile A_0 is also constant, *conservation of mass* requires that A, the cross-sectional area in the strained profile, be given by

$$A = A_0 \frac{ds}{dp}, \tag{2}$$

along the continuous strained profile.

In considering the *equilibrium requirements* of the cable we focus on the section of the cable lying between P_0 and P, when P lies in some general region of the cable demarked by P_n and

P_{n+1}. We take X to be the horizontal component of the reaction at the support P_0; Y_0 to be the vertical component of the reaction at the support P_0; and T to be the tension in the cable at P (Fig. 2). Clearly, because there is no horizontal loading on the cable, X represents the constant horizontal component of the cable tension.

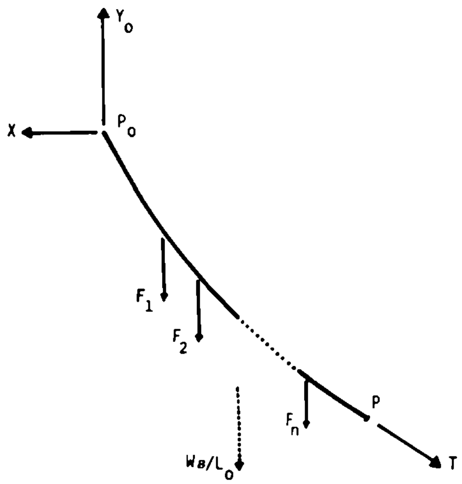

Fig. 2. Forces acting on a segment of the strained cable profile.

The forces are positive in the directions indicated in Fig. 2. Thus, resolving horizontally and vertically gives,

$$T\frac{\mathrm{d}x}{\mathrm{d}p} = X,$$

$$T\frac{\mathrm{d}y}{\mathrm{d}p} = Y_0 - \sum_{i=0}^{n} F_i - W\frac{s}{L_0}, \tag{3}$$

for the particle point P on the continuous strained profile, provided that we choose n such that P_n and P_{n+1} enclose $P(n = 0, 1, \ldots, N)$, and define $F_0 \equiv 0$.† In (3) the weight term is a consequence of the weight of the cable between the support P_0 and the particle point P being the same in both the unstrained and strained profiles. Preference is given to the weight term expressed in terms of the unstrained profile since the final objective of the present section is to derive solutions with s as their independent variable.

Moment equilibrium for the cable segment in Fig. 2 yields an expression which eqns (2), (3) ensure is satisfied automatically.

In establishing a *constitutive relation* for the cable we make the following assumptions: the only stress is the axial tensile stress which is uniform across a section of the cable (that is, the cable is flexible); and the strains are infinitesimal. Then, if E is Young's modulus for the cable (constant), we have, as a consistent specialization of the mathematical theory of elasticity,‡

$$T = EA_0\left(\frac{\mathrm{d}p}{\mathrm{d}s} - 1\right), \tag{4}$$

on the continuous strained profile.

The *end conditions* which hold at the cable supports P_0 and P_{N+1} are:

$$x = 0, \quad y = 0, \quad p = 0, \quad \text{at} \quad s = 0;$$
$$x = l, \quad y = h, \quad p = L, \quad \text{at} \quad s = L_0. \tag{5}$$

†This artificial null force is a device introduced to accommodate the full n-range in the summation ($n = 0, 1, \ldots, N$). Note that this full n-range is required if the general results are to include the case of the cable under self-weight alone ($N = 0$).

‡See, for example, Little [5] p. 38.

Here L is the length of the cable in the strained profile. Implicit in (5) is that x, y and p are functions of the single independent variable s.

Finally we have the *matching conditions* which ensure continuity of the cable at the points of application of the concentrated loads $P_n (n = 1, 2, \ldots, N)$, namely:

$$x_n^- = x_n^+, \quad y_n^- = y_n^+, \quad p_n^- = p_n^+, \quad \text{at} \quad s = s_n, \tag{6}$$

for $n = 1, 2, \ldots, N$. Here $x_n^- = \lim_{e \to 0} x(s_n - e), e > 0$, with analogous definitions for $x_n^+, y_n^-, \ldots, p_n^+$. Equations (6) complete the formulation of the general suspended cable problem we are currently concerned with.

We now proceed to the derivation of the parametric solution which completely describes the strained cable profile, that is, the determination of x, y, p, A and T as functions of the single independent variable s. First, the tension T.

In view of the geometric constraint (1), squaring the equilibrium requirements contained in (3) then adding the resulting expressions immediately provides, as the *solution for* $T = T(s)$,

$$T(s) = \sqrt{\left[\left(Y_0 - \sum_{i=0}^{n} F_i - W\frac{s}{L_0}\right)^2 + X^2\right]}, \tag{7}$$

on S, where $S = \{(s, n) | s_n < s < s_{n+1};\ n = 0, 1, \ldots, N\}$ with $s_0 = 0$.†

Next we treat the horizontal co-ordinate of the strained profile x. In deriving the *solution for* $x = x(s)$, we seek, as a preliminary to integration, an expression for dx/ds. To this end, note that $dx/ds = (dx/dp)(dp/ds)$. Observe further that dx/dp is given as a function of T in the first of the equilibrium requirements (3), while dp/ds may also be obtained as a function of T from the constitutive relation (4). Hence, on substituting for T from (7), we have

$$\frac{dx}{ds} = \frac{X}{EA_0} + \frac{X}{\sqrt{\left[\left(Y_0 - \sum_{i=0}^{n} F_i - W\frac{s}{L_0}\right)^2 + X^2\right]}}, \tag{8}$$

on S. Integrating (8) yields

$$x(s) = \frac{Xs}{EA_0} - \frac{XL_0}{W} \sinh^{-1}\left[\left(Y_0 - \sum_{i=0}^{n} F_i - W\frac{s}{L_0}\right) \Big/ X\right] + C_n, \tag{9}$$

on S, where $C_n (n = 0, 1, \ldots, N)$ are the constants of integration associated with the $N+1$ regions bounded by P_n, $P_{n+1} (n = 0, 1, \ldots, N)$, respectively. To evaluate the constants of integration in (9) we use the end and matching conditions. The first of the end conditions in (5), *viz.*, $x = 0$ at $s = 0$, in conjunction with (9), gives $C_0 = (XL_0/W) \sinh^{-1}(Y_0/X)$. The matching conditions on x in (6) furnish the following recurrence relation for C_n:

$$C_n = C_{n-1} + \frac{XL_0}{W}\left[\sinh^{-1}\left[\left(Y_0 - \sum_{i=0}^{n} F_i - W\frac{s_n}{L_0}\right) \Big/ X\right] - \sinh^{-1}\left[\left(Y_0 - \sum_{i=0}^{n-1} F_i - W\frac{s_n}{L_0}\right) \Big/ X\right]\right], \tag{10}$$

for $n = 1, 2, \ldots, N$. The value found for C_0 together with the recurrence relation (10) suggest a general expression for C_n which may then be readily verified by mathematical induction. Substitution of this general expression for C_n into (9) results in, as the solution for x,

$$\begin{aligned} x(s) = \frac{Xs}{EA_0} + \frac{XL_0}{W}\Bigg[\sinh^{-1}\frac{Y_0}{X} - \sinh^{-1}\left(\frac{Y_0}{X} - \sum_{i=0}^{n}\frac{F_i}{X} - \frac{Ws}{XL_0}\right) \\ + \sum_{i=0}^{n}\left(\sinh^{-1}\left[\frac{Y_0}{X} - \sum_{j=0}^{i}\frac{F_j}{X} - \frac{Ws_i}{XL_0}\right] - \sinh^{-1}\left[\frac{Y_0}{X} - \sum_{j=0}^{i}\frac{F_{j-1}}{X} - \frac{Ws_i}{XL_0}\right]\right)\Bigg]. \end{aligned} \tag{11}$$

†We use (a, b) to denote an ordered pair throughout. Thus S is the set of ordered pairs which formally defines the region of validity for (7), this region being previously described in (3) *et seq.*

As a consequence of satisfying the relevant matching conditions, eqn (11) holds on the extended domain $\bar{S} = \{(s, n)|s_n \le s \le s_{n+1};\ n = 0, 1, \ldots, N\}$ provided that we define $F_{-1} \equiv 0$ (recall that we have already defined $F_0 \equiv 0$, $s_0 \equiv 0$).

We now consider those aspects of the strained cable profile that are still outstanding, namely: the vertical co-ordinate y; the Lagrangian co-ordinate p; and the cross-sectional area A. Following an exactly analogous derivation to that employed for x yields the *solutions for* $y = y(s)$, $p = p(s)$. Eliminating dp/ds from the conservation of mass (2) and the constitutive relation (4), then introducing the solution found for the tension (7), yields the *solution for* $A = A(s)$. Thus we have the complete solution for this strained profile (in the interests of brevity the actual forms for y, p, and A are omitted at this juncture).

Note, however, that the expressions for T and x in (7) and (11) contain L_0, X, and Y_0—as do the expressions found for y, p, and A. With L_0 given, X and Y_0 are *unknown parameters* at this point. Thus we now establish a method of evaluating these parameters.

In deriving the solution for x, only the end condition at $s = 0$ in (5) is used. Similarly, in the derivations for y and p, only the end conditions at $s = 0$ are used. Hence on substituting $s = L_0$ in (11) and its counterparts for y and p and insisting on the satisfaction of the second set of end conditions in (5), we obtain three additional equations. The first of these equations (from $x = l$ at $s = L_0$) is a transcendental equation in X and Y_0. The second (from $y = h$ at $s = L_0$) is an algebraic equation in X and Y_0. These two equations constitute the requisite set for the complete determination of X and Y_0. The third equation (from $p = L$ at $s = L_0$) then serves to evaluate L, the strained length of the cable.

With a view to the concise exhibition of the solution and the set of equations for the determination of the unknown parameters, we now introduce the following *dimensionless* quantities:

$\xi = x/L_0$ horizontal co-ordinate of the strained profile;
$\eta = y/L_0$ vertical co-ordinate of the strained profile;
$\sigma = s/L_0$ Lagrangian co-ordinate of the unstrained profile;
$\sigma_n = s_n/L_0$ Lagrangian load co-ordinate ($n = 1, 2, \ldots, N$);
$\delta = h/L_0$ relative vertical displacement of the end-points;
$\gamma = l/L_0$ cable aspect ratio;
$\tau = T/W$ cable tension;
$\chi = X/W$ horizontal reaction at the supports P_0, P_{N+1};
$\phi = Y_0/W$ vertical reaction at the support P_0;
$\psi_n = F_n/W$ applied concentrated vertical loads ($n = 1, 2, \ldots, N$); and
$\beta = W/EA_0$ flexibility factor.

Note that in accordance with our definitions of s_0, F_{-1}, and F_0, extension of the above definitions leads to $\sigma_0 \equiv 0$, $\psi_{-1} \equiv 0$, and $\psi_0 \equiv 0$. We also introduce, for the partial sums of the dimensionless applied concentrated vertical loads ψ_n, the notation Ψ_n defined by $\Psi_n = \sum_{j=1}^{n} \psi_j$. Hence, $\Psi_{-1} \equiv 0$, $\Psi_0 \equiv 0$.

Under our dimensionless nomenclature we have, from (7), for the *dimensionless cable tension* $\tau = \tau(\sigma)$,

$$\tau(\sigma) = \sqrt{[(\phi - \Psi_n - \sigma)^2 + \chi^2]}. \tag{12}$$

Equation (12) holds on the domain $\Omega = \{(\sigma, n)|\sigma_n < \sigma < \sigma_{n+1};\ n = 0, 1, \ldots, N\}$. Further, for the *dimensionless horizontal and vertical co-ordinates of the strained profile*, $\xi = \xi(\sigma)$ *and* $\eta = \eta(\sigma)$, we have, from (11) and its counterpart for y,

$$\xi(\sigma) = \chi\Big[\beta\sigma + \sinh^{-1}\phi/\chi - \sinh^{-1}(\phi - \Psi_n - \sigma)/\chi$$
$$+ \sum_{i=0}^{n} (\sinh^{-1}[\phi - \Psi_i - \sigma_i]/\chi - \sinh^{-1}[\phi - \Psi_{i-1} - \sigma_i]/\chi)\Big],$$

$$\eta(\sigma) = \beta\sigma(\phi - \sigma/2) + \sqrt{(\phi^2 + \chi^2)} - \sqrt{[(\phi - \Psi_n - \sigma)^2 + \chi^2]}$$
$$+ \sum_{i=0}^{n} [\beta\psi_i(\sigma_i - \sigma) + \sqrt{[(\phi - \Psi_i - \sigma_i)^2 + \chi^2]} - \sqrt{[(\phi - \Psi_{i-1} - \sigma_i)^2 + \chi^2]}]. \quad (13)$$

Equations (13) hold on the extended domain $\bar{\Omega} = \{(\sigma, n)|\sigma_n \leq \sigma \leq \sigma_{n+1};\ n = 0, 1, \ldots, N\}$. The results contained in (12), (13) are those of most practical importance. The remainder of the solution (the strained Lagrangian co-ordinate and cross-sectional area) is set down in the Appendix. Finally we have, as the *dimensionless set of equations for the determination of the unknown parameters*,

$$\gamma/\chi - \beta = \sinh^{-1}\phi/\chi - \sinh^{-1}(\phi - \Psi_N - 1)/\chi$$
$$+ \sum_{i=0}^{N} [\sinh^{-1}(\phi - \Psi_i - \sigma_i)/\chi - \sinh^{-1}(\phi - \Psi_{i-1} - \sigma_i)/\chi],$$
$$\delta = \beta(\phi - 1/2) + \sqrt{(\phi^2 + \chi^2)} - \sqrt{[(\phi - \Psi_N - 1)^2 + \chi^2]}$$
$$+ \sum_{i=0}^{N} [\beta\psi_i(\sigma_i - 1) + \sqrt{[(\phi - \Psi_i - \sigma_i)^2 + \chi^2]} - \sqrt{[(\phi - \Psi_{i-1} - \sigma_i)^2 + \chi^2]}]. \quad (14)$$

With γ given, the unknown parameters in (14) are χ and ϕ.† The solution for these two parameters may be determined *via* a modified 2D Newton's method for any particular problem. Thus, eqns (12)–(14) constitute our solution for the static response of the elastic cable under the action of self-weight and N concentrated vertical loads.

The problem we have solved is nonlinear, the nonlinear effects stemming from the cable geometry. Our Lagrangian formulation is a natural one for this problem and this is reflected in the isolation of the nonlinearity in the single eqn (1), the geometric constraint. One might expect that, in other cable problems, such a formulation would again be appropriate.

Indeed, the present problem readily admits generalization to the case of non-vertical point loads and adaption to the case of distributed loadings. However, for the loaded "general" cable network, preliminary investigations indicate a mushrooming of the algebra to such an extent as to probably be prohibitive.

Specializing the solution contained in (12)–(14), by setting $N = 1$ generates the particular problem of the single point load.‡ In the next section, the solution for this problem is compared with a simple experiment.

2. THE CABLE UNDER A SINGLE LOAD: EXPERIMENTAL COMPARISON

The object of this section is to present an application of the general analysis of Section 1 to a specific problem. To this end a simple experiment is described and the attendant results displayed. Using the parametric values associated with the experiment, the theoretical counterpart is evaluated and compared.

The specific problem analysed involves a cable of deep profile with supports at the same level. Initially the cable hangs under its own weight. Subsequently the cable is loaded, in two equal load increments, at a point off-centre. The resulting displacements under all three loading systems are found.

For the experiment, a twisted-strand steel cable was supported in a relatively rigid test rig. At the supports, the cable was attached to yokes which allowed rotation in the vertical plane. The span between the supports (l) was 1·00 m and the relative vertical displacement of the supports (h) was zero. In the unstrained profile, the cable length (L_0) was 1·20 m and the cable cross-sectional area (A_0) was $1{\cdot}58 \times 10^{-6}$ m^2. Young's modulus for the cable (E) was 10^{11} Nm^{-2}.

The *actual self-weight* of the cable was so small as to be insufficient to straighten out kinks when the cable was suspended under its self-weight alone. These kinks indicated that the bending stresses in the cable were not negligible—contrary to one of the assumptions underlying the constitutive relation (4). To overcome this, twenty weights, each weighing 2·45 N, were attached symmetrically to the cable by means of small ferrules. The separation of the supports from the

†Alternatively, if χ is given instead of γ (as might occur when the cable is taut), eqns (14) serve to evaluate γ and ϕ. However, for this procedure a different non-dimensionalization would be more appropriate.

‡Specializing the solution by setting $N = 0$ generates the case of the cable under self-weight along—the elastic catenary.

nearest ferrules was 0·03 m (measured along the unstrained cable): elsewhere the separation of adjacent ferrules was 0·06 m. The total weight of the cable, twenty ferrules and twenty weights was 50 N. Consistent with the errors that accrued in other aspects of the experiment, this entire semi-discrete loading system was considered to be continuous and taken as the *effective self-weight* of the cable (W).

In addition to the effective self-weight, a concentrated vertical loading was applied in two successive increments of 9·8 N at the *load point* P_1, P_1 being 0·33 m from the support P_0 (measured along the unstrained cable). Thus the independent dimensionless parameters for the experiment were as given in Table 1.

Table 1. Specific values of dimensionless cable parameters

Cable geometry:	$\gamma = 0.833$
	$\delta = 0$
Flexibility factor:	$\beta = 3.16 \times 10^{-4}$
Cable loadings at:	$\sigma_1 = 0.275$
- after first increment	$\psi_1 = 0.196$
- after second increment	$\psi_1 = 0.392$

The recording of the experiment was accomplished by the superposition of multiple exposures on a single plate using a large format camera. To enhance the contrast of the resulting photograph, the cable was painted white while the ferrules, weights, and background were painted black. White dots located the fixed end points of the cable and the points in the various strained profiles corresponding to the load point $P_1(\sigma_1 = 0{\cdot}275)$. The first exposure was taken when the cable hung under its effective self-weight; the second when $\psi_1 = 0{\cdot}196$; and the third when $\psi_1 = 0{\cdot}392$ (Fig. 3A). As one would have anticipated, increases in the applied loads lowered the load point and in Fig. 3A the white strained profiles are accordingly associated with this increasing sequence of loadings.

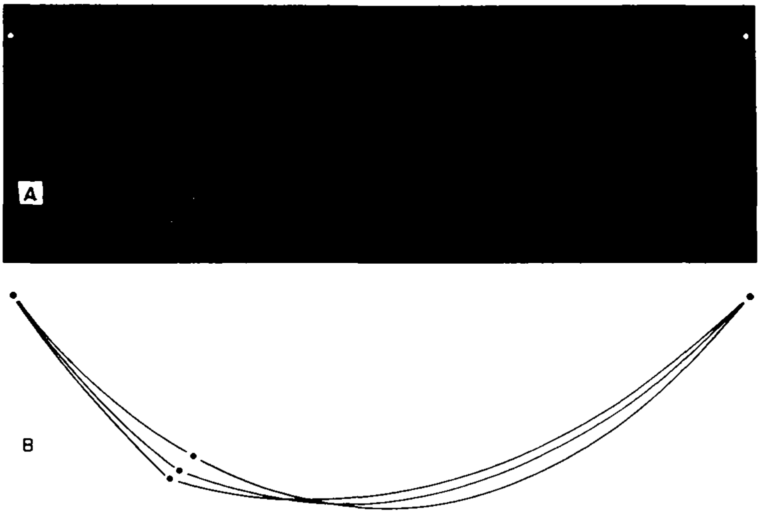

Fig. 3. Comparison of experimental and theoretical results for the cable under the action of a single concentrated load: (A) experimental; (B) theoretical.

The cartesian co-ordinates of P_1 in these three strained profiles were measured directly from the photograph in Fig. 3A. The values so obtained are listed, in their respective dimensionless forms, under the experiment columns of Table 2.

The corresponding theoretical analysis of the problem is performed in the following manner. First we consider the cable under effective self-weight alone. Thus N equals zero in eqns (13) and (14). Inspection of the second of (14) immediately reveals that ϕ, the vertical reaction at the

Table 2. Dimensionless cartesian co-ordinates of the load point in the three strained profiles

Load Case	Horizontal co-ordinate $\xi(0.275)$		Vertical co-ordinate $\eta(0.275)$	
	Experiment	Theory	Experiment	Theory
Effective self-weight	0.208	0.202	0.189	0.184
Effective self-weight plus single load increment (ψ_1 = 0.196)	0.193	0.186	0.208	0.201
Effective self-weight plus double load increment (ψ_1 = 0.392)	0.182	0.176	0.215	0.211

support P_0, equals 0·5 in this instance.† Substituting the values of γ, δ, and β contained in Table 1 then furnishes a single transcendental equation for the horizontal reaction χ. For the present problem, the cable has a large sag so that one would expect χ to be approximately equal to ϕ. Using this approximation and taking χ equal to 0·5 as an initial estimate, the transcendental equation for χ can be solved using a modified 1D Newton's method. Convergence is rapid and χ is found to be 0·390. This value, conjunction with the ϕ value and the γ, δ, β values of Table 1, is then substituted in (13) (with $N = 0$) to yield (ξ, η) co-ordinate pairs on the strained profile as the Lagrangian co-ordinate σ is varied. These co-ordinate pairs form the basis of the curves in Fig. 3B. Here, black lines indicate the cable profiles with black dots locating the fixed end points of the cable and the points in the various strained profiles corresponding to the load point P_1. The curve passing through the uppermost load point represents the profile for the cable under effective self-weight alone. The particular co-ordinate pair for the load point is recorded under the theoretical columns of Table 2.

Turning to the cable under the first increment of concentrated load, we set N equal to one in (13), (14). Equations (14), with ψ_1 put equal to 0·196 and γ, δ, β and σ_1 taking on the values in Table 1, then provide two equations for the determination of the reactions, χ and ϕ. Since the additional concentrated load is not large in comparison with the effective self-weight, the values for χ and ϕ found for the cable in the absence of the load increment constitute suitable initial estimates. With these initial estimates, a modified 2D Newton's method gives χ and ϕ as 0·480 and 0·662 respectively—convergence again being rapid. With the reactions at the support P_0 now evaluated, the (ξ, η) co-ordinate pairs can be calculated as previously. The curve so defined is shown in Fig. 3B as passing through the middle load point: the actual co-ordinate pair for the load point is given in Table 2.

Finally, with the second equal increment of concentrated load acting, a completely analogous procedure yields the reactions (χ = 0·571 and ϕ = 0·827), the pertinent curve in Fig. 3B (passing through the lowest load point), and the accompanying co-ordinate pair for the load point in Table 2.

In Fig. 3, the good agreement between the theoretical curves and the corresponding experimental profiles is apparent. This agreement is confirmed by the particular results for the load point in Table 2, wherein the maximum discrepancy between theoretical and experimental values of ξ, η is less than 4% (this being within experimental error). Further, both sets of results in Table 2 illustrate the geometric nonlinear response of the cable—this feature is, of course, evident in Fig. 3. Also, for both sets of results, the horizontal and vertical displacements of the load point are of the same order of magnitude—this is a consequence of the large sag of the cable analysed and the positioning of the load point about a quarter of the way along the cable. Lastly, we remark that elastic effects are insignificant in this particular problem owing to the large sag and the relatively inextensible nature of the cable material.

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†Clearly, from the symmetry of the problem, this must be the case when no concentrated loads are applied and the cable supports are level.

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APPENDIX

The solution for the strained cable length and cross-sectional area

Here, for completeness, we present those remaining aspects of the solution contained in (12)–(14), namely: the results for the Lagrangian co-ordinate in the strained profile p; and the cross-sectional area in the strained profile A. To this end, the following additional dimensionless quantities are introduced:

$\rho = p/L_0$ Lagrangian co-ordinate of the strained profile;
$\alpha = A/A_0$ cross-sectional area of the strained profile; and
$\lambda = L/L_0$ cable stretch factor.

Using this nomenclature the results are:

$$\begin{aligned}\rho(\sigma) = \sigma + \frac{\beta}{2}\Big[&\phi\sqrt{(\phi^2+\chi^2)} - (\phi - \Psi_n - \sigma)\sqrt{[(\phi-\Psi_n-\sigma)^2+\chi^2]}\\ &+ \chi^2(\sinh^{-1}\phi/\chi - \sinh^{-1}[\phi-\Psi_n-\sigma]/\chi)\\ &+ \sum_{i=0}^{n}([\phi-\Psi_i-\sigma_i]\sqrt{([\phi-\Psi_i-\sigma_i]^2+\chi^2)} - [\phi-\Psi_{i-1}-\sigma_i]\sqrt{([\phi-\Psi_{i-1}-\sigma_i]^2+\chi^2)}\\ &+ \chi^2[\sinh^{-1}(\phi-\Psi_i-\sigma_i)/\chi - \sinh^{-1}(\phi-\Psi_{i-1}-\sigma_i)/\chi])\Big],\end{aligned}$$

$$\alpha(\sigma) = 1/[1+\beta\sqrt{[(\phi-\Psi_n-\sigma)^2+\chi^2]}],$$

holding on $\bar{\Omega}$ and Ω, respectively. The last of our end conditions (5) ($p = L$ at $s = L_0$) together with the first of the above then furnishes an expression for the stretch factor λ. The complete solution now achieved may be verified directly by substitution into the formulating eqns (1) through (6).